## Uncertainty Principle in Gaussian wave packets:-

Gaussian wave function

$$\psi(x) = \underbrace{\left(\frac{2}{\pi a^2}\right)^{1/4}}_{\text{constant}} \underbrace{e^{-(x-x_0)^2/a^2}}_{\text{G. Wave}} \underbrace{e^{\frac{ip_0 x}{\hbar}}}_{\text{oscillatory function}}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^*_{(x)} x \, \psi_{(x)} \, dx$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 \, dx$$

$$|\psi(x)|^2 = \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2(x-x_0)^2/a^2}$$

$$\langle x\rangle = \int_{-\infty}^{\infty}(x_0 - x_0 + x)\left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2(x-x_0)^2/a^2} \, dx$$

$$\langle x\rangle = \left(\frac{2}{\pi a^2}\right)^{1/2} \underbrace{\int (x-x_0)\, e^{-2(x-x_0)^2/a^2} dx}_{\substack{\text{function is odd.} \\ \text{Therefore it will be zero}}} + x_0\left(\frac{2}{\pi a^2}\right)^{1/2}\int_{-\infty} e^{-2(x-x_0)^2/a^2} dx$$

$$\langle x\rangle = 0 + x_0\left(\frac{2}{\pi a^2}\right)^{1/2}\int_{-\infty}^{\infty} e^{-2(x-x_0)^2/a^2} dx$$

$$\langle x\rangle = x_0 \left(\frac{2}{\pi a^2}\right)^{1/2} \sqrt{\frac{\pi \times a^2}{2}} \qquad \left(\because \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}\right)$$

$$\langle x\rangle = x_0 \left(\frac{\cancel{2}}{\cancel{\pi a^2}} \times \frac{\cancel{\pi a^2}}{\cancel{2}}\right)^{1/2}$$

$$\boxed{\langle x\rangle = x_0}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2(x-x_0)^2/a^2} dx$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} (x-x_0)^2 \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2(x-x_0)^2/a^2} dx - x_0^2 \left(\frac{2}{\pi a^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-2(x-x_0)^2/a^2} dx$$
$$+ 2x_0 \int_{-\infty}^{\infty} x \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2(x-x_0)^2/a^2} dx$$

$$\langle x^2 \rangle = \left(\frac{2}{\pi a^2}\right)^{1/2} \frac{\sqrt{\pi}\cdot 1}{2^1 \left(\frac{2}{a^2}\right)^{1+1/2}} - x_0^2 \left(\frac{2}{\pi a^2}\right)^{1/2} \sqrt{\frac{\pi a^2}{2}} + 2x_0 \times x_0 \left(\frac{2}{\pi a^2}\right)^{1/2} \sqrt{\frac{\pi a^2}{2}}$$

$$\langle x^2 \rangle = \left(\frac{2\cancel{\pi}}{\cancel{\pi} a^2}\right)^{1/2} \frac{1}{2}\left(\frac{a^2}{2}\right)^{3/2} - x_0^2 \left(\frac{\cancel{2}}{\cancel{\pi a^2}} \times \frac{\cancel{\pi a^2}}{\cancel{2}}\right)^{1/2} + 2x_0^2 \left(\frac{\cancel{2}}{\cancel{\pi a^2}} \times \frac{\cancel{\pi a^2}}{\cancel{2}}\right)^{1/2}$$

$$\langle x^2 \rangle = \frac{1}{2}\left(\frac{2}{a^2}\right)^{1/2} \frac{a^2}{2}\left(\frac{a^2}{2}\right)^{1/2} - x_0^2 + 2x_0^2$$

$$\langle x^2 \rangle = \frac{1}{2}\left(\frac{\cancel{2}}{\cancel{a^2}} \times \frac{\cancel{a^2}}{\cancel{2}}\right)^{1/2} \frac{a^2}{2} + x_0^2$$

$$\langle x^2 \rangle = \frac{a^2}{4} + x_0^2$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta x = \sqrt{\frac{a^2}{4} + \cancel{x_0^2} - \cancel{x_0^2}}$$

$$\Delta x = \sqrt{\frac{a^2}{4}}$$

$$\boxed{\Delta x = \frac{a}{2}} \qquad \text{——— (1)}$$

Now

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}} \psi(x)\, dx$$

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2}{\pi a^2}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/a^2} \, e^{-\frac{ipx}{\hbar}} \, e^{\frac{ip_0x}{\hbar}} \, dx$$

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2}{\pi a^2}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/a^2} \, e^{-i(p-p_0)\frac{x}{\hbar}} \, dx$$

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2}{\pi a^2}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/a^2} \left[\cos\left(\frac{p-p_0}{\hbar}\right)x - \underbrace{i\sin\frac{(p-p_0)x}{\hbar}}_{\text{odd}}\right] dx$$

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2}{\pi a^2}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/a^2} \cos\left(\frac{p-p_0}{\hbar}\right)x \, dx \quad - 0$$

$$\boxed{\psi(p) = \left(\frac{a^2}{2\pi}\right)^{1/4} e^{-\frac{a^2(p-p_0)^2}{4\hbar^2}}}$$

Centre of the Gaussian wave function is $p_0$

Therefore $\langle p \rangle = p_0$ (similar to the x)

$$\langle p^2 \rangle = \frac{\text{Denominator}}{4} + (\text{Centre})^2$$

$$\langle p^2 \rangle = \frac{\cancel{4}\hbar^2}{\cancel{4}a^2} + p_0^2$$

$$\langle p^2 \rangle = \frac{\hbar^2}{a^2} + p_0^2$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$\Delta p = \sqrt{\frac{\hbar^2}{a^2} + \cancel{p_0^2} - \cancel{p_0^2}}$$

$$\boxed{\Delta p = \frac{\hbar}{a}} \quad \text{—— (II)}$$

From equation (I) and (II)

$$\Delta x \, \Delta p = \frac{\cancel{a}}{2} \times \frac{\hbar}{\cancel{a}}$$

$$\boxed{\Delta x \, \Delta p = \frac{\hbar}{2}}$$

Uncertainty Principle for Gaussian wave Packet